THE EFFECT OF INJECTION ON THE SUPERSONIC STREAMLINING OF A CONE

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We present an approximate calculation for the parameters of the flow field behind a conic shock wave in the presence of injection perpendicular to the surface of a cone and with hypersonic flow in the external stream.

Aroesty and Davis [1] give an exact solution for the equation of motion for a nonviscous gas, with provision made for the effect of surface mass transfer. The problem is considered for the symmetric supersonic streamlining of a cone of infinite length, under the condition of injection perpendicular to the surface, a constant wall temperature, and a uniform flow rate for the injected gas. The cited solution reduces to the determination of the surface separating the external potential flow, passing through the shock wave, from the internal potential flow of the injected gas.

This article demonstrates how to obtain the relationship between the angle of inclination for the compression shock and the geometry of the cone from the solution given in [1] for an external flow at hypersonic velocities, how to determine the detachment of the shock from the apex of the cone, and also how to make provision for the effect of mass transfer on the coefficient of flow rate and surface drag for the external flow.

The boundary separating the external from the internal potential flow (Fig. 1) is determined [1] by the relationship

$$\ln\frac{1+\mu_s}{1-\mu_s}+\frac{2\mu_s}{1-\mu_s^2}=\ln\frac{1+\mu_c}{1-\mu_c}-\frac{2}{\mu_c}, \tag{1}$$

where $\mu = \cos\theta$.

With a series expansion in $\cos\theta_s$ in (1) and neglecting terms of fifth and higher orders, we find

$$\frac{4}{3}\mu_s^3-A\mu_s^2-4\mu_s+A=0, \tag{2}$$

where

$$A=\ln\frac{1+\mu_c}{1-\mu_c}-\frac{2}{\mu_c}.$$

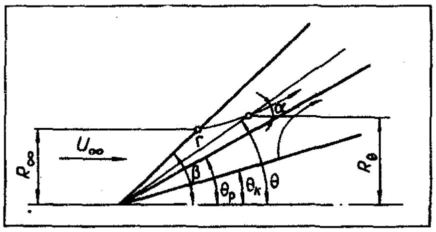

Fig. 1. Diagram showing the streamlining of a cone by a supersonic flow of gas and with injection normal to the wall surface.

One of the roots of Eq. (2) $\theta_c > 0$ yields $\mu_s > 1$; for $\theta_c > 0$ the second root yields $\mu_s > |1|$, and we therefore drop these two roots, as physically nonreal. Finally, we have

$$\theta_s=\cos^{-1}\left[\frac{A}{4}-\frac{(16+A^2)^{\frac{1}{2}}}{2}\cos\left(60^\circ+\frac{1}{3}\cos^{-1}\frac{A^3}{(16+A^2)^{3/2}}\right)\right]. \tag{3}$$

In comparison with the exact equation (1) this expression yields an error of no more than 1.5% in the calculation of θ_s. It should be noted that the numerical integration of the equation of motion that was performed in [1, 2] for various values of the cone angles and various Mach numbers of subsonic injection demonstrated

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the weak sensitivity of the angle θ with respect to the effect of compressibility in the internal flow zone.

Let us apply the approximate expression (3) to the problem of subsonic injection into a hypersonic flow. Let us examine the hypersonic flow in an internal layer of constant density. Since it is agreed that the flow is axisymmetric and isentropic, it is easy to derive the following from the expression for the velocity potential [3] and for the boundary condition $du/d\theta = v = 0$ at the separation surface:

$$\frac{\varepsilon}{1-\varepsilon} = \mu_\beta(1-\mu_\beta^2)\left[\frac{1}{2}\ln\frac{(1+\mu_c)(1-\mu_\beta)}{(1-\mu_c)(1+\mu_\beta)} - \frac{1}{\mu_c} - \frac{\mu_\beta}{1-\mu_\beta^2}\right].$$

Correspondingly, for the pressure gradient in the external flow zone we find

$$\frac{d\left(\frac{c_s}{2}\right)}{d\mu} = \frac{(1-\varepsilon)^2\mu_\beta^2(1-\mu_\beta^2)^2}{\varepsilon(1-\mu^2)} \times \left[\frac{1}{2}\ln\frac{(1+\mu_c)(1-\mu_\beta)}{(1-\mu_c)(1+\mu_\beta)} - \frac{1}{\mu_c} - \frac{\mu_\beta}{1-\mu_\beta^2}\right]. \quad (5)$$

Since

$$\frac{1}{2}\ln\frac{(1+\mu_c)(1-\mu_\beta)}{(1-\mu_c)(1+\mu_\beta)} - \frac{1}{\mu_c} - \frac{\mu_\beta}{1-\mu_\beta^2} = Q(\mu_s) - Q(\mu_\beta),$$

$$Q(\mu) = \frac{1}{2}\ln\frac{1+\mu}{1-\mu} + \frac{\mu}{1-\mu^2},$$

the difference $Q(\mu_s) - Q(\mu_\beta)$ for a thin layer of the external flow can be expanded in a Taylor series in the vicinity of μ_β and we can obtain the following result:

$$\mu_\beta(1-\mu_\beta^2)\left[\frac{1}{2}\ln\frac{(1+\mu_c)(1-\mu_\beta)}{(1-\mu_c)(1+\mu_\beta)} - \frac{1}{\mu_c} - \frac{\mu_\beta}{1-\mu_\beta^2}\right] = 2\xi + \xi^2 + 0(\xi^3), \quad (6)$$

where

$$\xi = \frac{\cos\beta\sin(\beta-\theta_s)}{\sin\theta_s}$$

Expression (6) permits us to satisfy condition (4) by means of the approximate equation

$$\frac{0.5\varepsilon}{1-0.5\varepsilon} = \left\{\cos\beta\sin\left\{\beta - \arccos\left[\frac{A}{4} - \frac{(16+A^2)^{\frac{1}{2}}}{2}\cos\left(60^\circ + \frac{1}{3}\arccos\frac{A^3}{(16+A^2)^{3/2}}\right)\right]\right\}\right\}$$
$$\times\left\{\sqrt{1-\left[\frac{A}{4} - \frac{(16+A^2)^{\frac{1}{2}}}{2}\cos\left(60^\circ + \frac{1}{3}\arccos\frac{A^3}{(16+A^2)^{3/2}}\right)\right]^2}\right\}^{-1}. \quad (7)$$

When the conditions in the oncoming stream are specified and ε is defined as a function of β and M_∞, Eq. (7) establishes an implicit relationship between the semivertex angle of the compression shock and the semivertex angle of the cone.

Let us use the results from the analysis in [4] of the independence from M_∞ of the coefficient for the pressure behind the conic shock wave for our case of the streamlining of the cone. From Eqs. (3) and (7) we will then obtain an analytical expression for the angle of shock inclination as a function of θ_c in explit form:

$$\sin^2\beta = \frac{1}{M_\infty^2} + 0.038\left\{10^{-1}\arccos\left[\frac{1}{4}\ln\frac{1+\mu_c}{1-\mu_c} - \frac{1}{2\mu_c} - \sqrt{4+\frac{1}{4}\left(\ln\frac{1+\mu_c}{1-\mu_c} - \frac{2}{\mu_c}\right)^2}\right.\right.$$
$$\left.\left.\times\cos\left(60^\circ + \frac{1}{3}\arccos\frac{\left[\ln\frac{1+\mu_c}{1-\mu_c} - \frac{2}{\mu_c}\right]^3}{\left[16+\left(\ln\frac{1+\mu_c}{1-\mu_c} - \frac{2}{\mu_c}\right)^2\right]^{3/2}}\right)\right]\right\}^{1.87}. \quad (8)$$

Equation (8) makes it possible substantially to simplify the calculation of the flow-field parameters behind the conic shock wave in the presence of injection perpendicular to the cone surface.

From the approximate relationship

$$\beta - \theta_s = \operatorname{tg}\beta\left[\frac{1}{(k+1)fM_\infty^2} + \frac{k-1}{2(k+1)}\right]$$

and from (5) and (8) we define the pressure at the separation surface in the form

$$p_{\theta_s} = \rho_\infty U_\infty^2\left[f - \frac{3}{2(k+1)M_\infty^2} - \frac{3f(k-1)}{4(k+1)}\right] + p_\infty, \tag{9}$$

where $f = \sin^2\beta$ is found from (8).

We will also show the relationship which determines the detachment of the compressor shock from the apex of the cone. With regard to hypersonic flow in the case of a constant density in the external layer and with subsonic normal injection, we have

$$\operatorname{tg}\beta = \left[\frac{1}{(k+1)fM_\infty^2} + \frac{k-1}{2(k+1)}\right]^{-0.5}\psi(\eta), \tag{10}$$

where

$$\psi(\eta) = \frac{2\eta}{1 \pm \sqrt{1-4\eta^2}}, \tag{11}$$

$$\eta = \left[\frac{1}{(k+1)fM_\infty^2} + \frac{k-1}{2(k+1)}\right]^{0.5}\left[1 - \frac{1}{(k+1)fM_\infty^2} - \frac{k-1}{2(k+1)}\right]^{-1}$$
$$\times \operatorname{tg}\left\{\arccos\left[\frac{A}{4} - \frac{(16+A^2)^{\frac{1}{2}}}{2}\cos\left(60^\circ + \frac{1}{3}\arccos\frac{A^2}{(16+A^2)^{3/2}}\right)\right]\right\}. \tag{12}$$

The plus sign in (11) corresponds to a solution with a weak shock, while the minus sign corresponds to a solution with a strong shock. A solution with an attached shock is possible up to the maximum value for the cone angle, which is determined for $\psi(\eta) = 1$. Therefore,

$$\operatorname{tg}\left\{\arccos\left[\frac{A}{4} - \frac{(16+A^2)^{\frac{1}{2}}}{2}\cos\left(60^\circ + \frac{1}{3}\arccos\frac{A^3}{(16+A^2)^{3/2}}\right)\right]\right\}_{\text{det}}$$
$$= \frac{1}{2}\left[1 - \frac{1}{(k+1)fM_\infty^2} - \frac{k-1}{2(k+1)}\right]\left[\frac{1}{(k+1)fM_\infty^2} + \frac{k-1}{(k+1)2}\right]^{-0.5}. \tag{13}$$

The cone angle corresponding to the detachment of the shock for a known value of M_∞, with this equation is found through a trial and error procedure.

Approximation of a flow with constant density in the layer of the external flow behind the shock wave yields the following value for the flow rate and the pressure over the stream surface of the conic flow [5]:

$$\varphi = \frac{R_\infty^2}{R_\theta^2} = \frac{c_1 + c_2\left(\frac{1}{2}\ln\frac{1+\mu}{1-\mu} + \frac{\mu}{1-\mu^2}\right)_\theta}{c_1 + c_2\left(\frac{1}{2}\ln\frac{1+\mu}{1-\mu} + \frac{\mu}{1-\mu^2}\right)_\beta}, \tag{14}$$

$$\frac{c_x}{2} = \frac{X}{\rho_\infty U_\infty^2 F} = \varphi\left[\cos\alpha\sqrt{\left(c_1 + \frac{c_2}{2}\ln\frac{1+\mu}{1-\mu}\right)_\theta^2 + c_2^2\left(1 + \frac{\mu^2}{1-\mu^2}\right)_\theta} - 1\right] + \frac{\bar{p}}{2}, \tag{15}$$

where

$$F = \pi R_\theta^2,$$

$$\alpha = \theta + \operatorname{arctg}\frac{\sqrt{1-\mu_\theta^2}\left[c_1 + c_2\left(\frac{1}{2}\ln\frac{1+\mu}{1-\mu} + \frac{\mu}{1-\mu^2}\right)_\theta\right]}{c_1\mu_\theta + c_2\left(\frac{\mu}{2}\ln\frac{1+\mu}{1-\mu} - 1\right)_\theta},$$

$\bar{p}$ is a dimensionless coefficient of the pressure on the conic surface whose semivertex angle is θ; c_1 and c_2 are constants determined from the conditions at the shock.

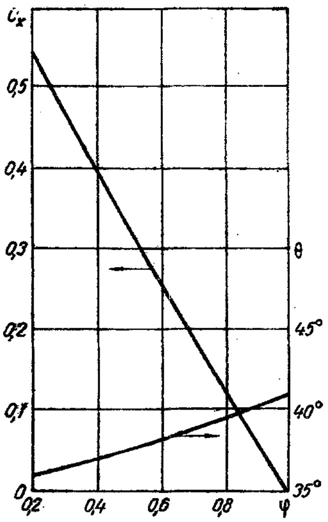

Fig. 2. The resistance factor c_x for the flow surface as a function of the flow-rate φ and as a function of the flow-rate factor and the flow-rate factor φ in the external flow zone as a function of the angle θ.

For the problem of hypersonic streamlining of a cone with injection, we have

$$c_1 = \frac{2}{(k+1)fM_\infty^2} + \frac{k-1}{k+1} + \frac{2f(1-f)^{\frac{1}{2}}}{k+1}\left(1 - \frac{1}{fM_\infty^2}\right)\left[\frac{1}{2}\ln\frac{1+(1-f)^{\frac{1}{2}}}{1-(1-f)^{\frac{1}{2}}} + \frac{(1-f)^{\frac{1}{2}}}{f}\right], \tag{16}$$

$$c_2 = -\frac{2f}{k+1}\left(1 - \frac{1}{fM_\infty^2}\right)(1-f)^{\frac{1}{2}}, \tag{17}$$

$$\frac{\bar{p}}{2} = \frac{2f}{k+1}\left(1 - \frac{1}{fM_\infty^2}\right) + (1-f)(\arcsin\sqrt{f} - \theta)(\arcsin\sqrt{f} + \theta - 2\theta_K)\left[\frac{k-1}{k+1} + \frac{2}{(k+1)fM_\infty^2}\right]^{-1}. \tag{18}$$

The values of θ and θ_s in (18) are taken in radians.

As an example, we have calculated the coefficients c_x and φ in the flow for the supersonic streamlining of a circular cone and for the case of normal injection through the wall. The calculation was carried out with (14) and (15) for $M_\infty = 5.2$ and $\theta = 1°8'5''$ and this is shown by the curves in Fig. 2.

NOTATION

U_∞	is the velocity of the unperturbed flow;
β	is the angle between the direction of the shock wave and the unperturbed flow;
θ_c	is the semivertex angle of the cone;
θ_s	is the semivertex angle of the surface separating the layer of injected gas from the external flow;
v and u	are the velocity components in the directions θ and r;
θ and r	are polar coordinates;
$\varepsilon = \rho_\infty/\rho_\beta$	is the ratio of the densities in front of and behind the shock wave;
$k = c_p/c_v$.	

Subscripts

θ, β, ∞, s, and c correspond to the conditions on the ray θ, immediately behind the shock wave, in the unperturbed flow, at the boundary of separation, and on the cone surface.

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